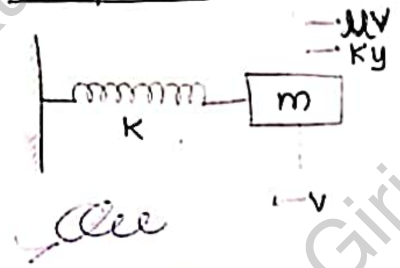
Damped vibration:



Let us consider a mass 'm' attached to spring with a force constant K. When external force is applied to the mass along the length of the spring. It starts vibrating & after some time its vibration becomes zero because of frictional force (dissipative force) & restoring force of spring the vibration becomes zero.

$$F_{net} = ma \qquad (4)$$

$$-\mu v - Ky = ma$$

$$ma + \mu v + Ky = 0$$

$$m\frac{d^2y}{dt^2} + \mu\frac{dy}{dt} + Ky = 0 \quad \text{--- (1)}$$

$$\frac{d^2y}{dt^2} + \frac{\mu}{m}\frac{dy}{dt} + \frac{K}{m}y = 0 \quad \text{--- (2)}$$

$$\left\{ \text{where, } 2b = \frac{\mu}{m} \ \& \ \omega^2 = \frac{K}{m} \right\}$$

$$\frac{d^2y}{dt^2} + 2b\frac{dy}{dt} + \omega^2 y = 0 \quad \cdots\cdots (1)$$

$$(D^2 + 2bD + \omega^2)y = 0$$

$$y = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y = \frac{-2b \pm \sqrt{(2b)^2 - 4(1)(\omega^2)}}{2(1)}$$

$$y = \frac{-2b \pm \sqrt{4b^2 - 4\omega^2}}{2}$$

$$y = \frac{-\cancel{2}b \pm \cancel{2}\sqrt{b^2 - \omega^2}}{\cancel{2}}$$

$$y = -b \pm \sqrt{b^2 - \omega^2}$$

$$y_1 = -b + \sqrt{b^2 - \omega^2}$$

$$y_2 = -b - \sqrt{b^2 - \omega^2} \qquad \text{where, } b = \frac{\mu}{2m}$$

BASIC

1. If roots are different i.e m_1 & m_2 then soln. $y = c_1 e^{m_1 t} \pm c_2 e^{m_2 t}$
2. If $m_1 = m_2 = m$ $\quad y = (c_1 + tc_2)e^{mt}$ $\quad$ { C is constant it may be A, B, C. }
3. If $\alpha \pm i\beta$ $\quad y = e^{\alpha t}(\cos\beta t \pm \sin\beta t)$

$y = Ae^{(-b+\sqrt{b^2-\omega^2})t} + Be^{(\ldots)t}$ ------ ②

Eqⁿ ② is solution of eqⁿ ①

$$\boxed{y = ae^{-bt}\sin(\omega t - \alpha)} \text{ ------ ③}$$

→ Soln of eqⁿ ①

$$\omega_1^2 = \omega^2 - b^2$$

$$\omega_1 = \sqrt{\omega^2 - b^2}$$

$$\boxed{\omega_1 = \sqrt{\frac{k}{m} - \frac{\mu^2}{4 m^2}}}$$ → angular frequency

$$f_1 = \frac{\omega_1}{2\pi}$$

$$f_1 = \frac{1}{2\lambda}$$

$y = ae^{-bt}\sin\omega_1 t$

$y' = a\omega_1 \ldots$

$y'' = \ldots$

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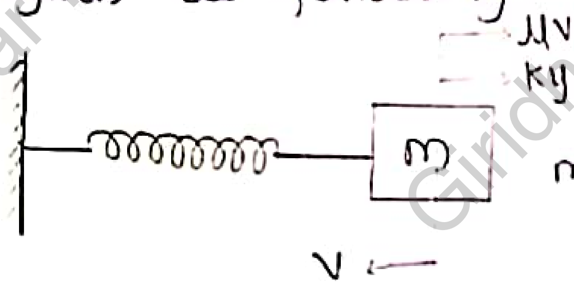
Forced vibration :

Let us consider a body under free vibration. after some time the body comes to rest because of damping forces namely restoring force and frictional force of air.

If external periodic force is applied to the body then body gradually gains the frequency of external periodic force.

For the forced vibration the eqⁿ can be given as following :



$$F_{net} = ma$$

$$F\sin pt - \mu v - ky = ma$$

$$m\frac{d^2y}{dt^2} + \mu\frac{dy}{dt} + ky = F\sin pt \quad ①$$

$F_e = F\sin pt$

Here 'p' is frequency of external periodic force. The particular solution of eqⁿ ① is

$$y = a\sin(pt - \alpha) \text{ ---- ②}$$

Now, differentiate eqⁿ ② w.r.t time

$$\frac{dy}{dt} = \frac{d}{dt}[a\sin(pt-\alpha)]$$

$$dy/dt = a\cos(pt-\alpha).p$$

$$dy/dt = ap\cos(pt-\alpha) \text{ --- ③}$$

(6) Again differentiate eq^n (5) w.r.t time.

$$\frac{d^2y}{dt^2} = -ap^2\sin(pt-\alpha) \quad \ldots\ldots (11)$$

Substituting eq^n (2), (3) & (4) in (1)

$$-map^2\sin(pt-\alpha) + \mu ap\cos(pt-\alpha) + ka\sin(pt-\alpha) = F\sin pt \quad (5)$$

$$-map^2(\sin pt\cos\alpha - \cos pt\sin\alpha) + \mu ap[\cos pt\cos\alpha + \sin pt\sin\alpha] + ka(\sin pt\cos\alpha - \cos pt\sin\alpha) = F\sin pt \quad \ldots (6)$$

If $\sin pt = 1$, $\cos pt = 0$ put in (6)

$$\cancel{-map^2[(1)\cos\alpha - (0)(1)] + \mu ap[(0)}$$

$$-map^2[\cos\alpha - 0] + \mu ap[\sin\alpha] + ka[\cos\alpha) = F$$

$$-map^2(\cos\alpha) + \mu ap(\sin\alpha) + ka(\cos\alpha) - F = 0 \quad \ldots (7)$$

If $\sin pt = 0$, $\cos pt = 1$ put in (6)

$$\cancel{-map^2(1) + \mu ap(0) + ka(1) - F = 0}$$

$$\cancel{-map^2 + ka - F = 0}$$

$$-map^2(-\sin\alpha) + \mu ap(\cos\alpha) + ka(-\sin\alpha) = 0$$

$$map^2\sin\alpha + \mu ap\cos\alpha - ka\sin\alpha = 0 \quad \ldots\ldots (8)$$

dividing eq^n (8) by $\cos\alpha$

$$map^2\tan\alpha + \mu ap - ka\tan\alpha = 0$$

$$a(mp^2\tan\alpha + \mu p - k\tan\alpha) = 0$$

$$mp^2\tan\alpha + \mu p - k\tan\alpha = 0$$

$$\tan\alpha(mp^2 - k) + \mu p = 0$$

$$-\tan\alpha(k - mp^2) + \mu p = 0$$

$$\tan\alpha = \frac{(\mu p)}{(k - mp^2)} = \frac{A}{B}$$

where, $A = \mu p$

$B = k - mp^2$

$$\therefore \sin\alpha = \frac{A}{\sqrt{A^2+B^2}} \quad \text{and} \quad \cos\alpha = \frac{B}{\sqrt{A^2+B^2}}$$

Now dividing eq^n (7) by $\cos\alpha$

(7) $-map^2 + \mu ap\tan\alpha + ka = \frac{F}{\cos\alpha}$

$$a(-mp^2 + \mu p\tan\alpha + k) = \frac{F}{\cos\alpha}$$

$$a[(k-mp^2) + \mu p\tan\alpha] = \frac{F}{\cos\alpha}$$

$$a\left[B + A\frac{A}{B}\right] = \frac{F\sqrt{A^2+B^2}}{B}$$

$$a\left[\frac{B^2+A^2}{\cancel{B}}\right] = \frac{F\sqrt{A^2+B^2}}{\cancel{B}}$$

$$a = \frac{F\sqrt{A^2+B^2}}{A^2+B^2}$$

$$a = \frac{F}{\sqrt{A^2+B^2}} \text{ ------- } (9)$$

Substitute eq^n (9) in (2)

$$y = \frac{F}{\sqrt{A^2+B^2}}\sin(pt-\alpha)$$

$$y = \frac{F}{\sqrt{\mu^2p^2 + (k-mp^2)^2}}\sin(pt-\alpha) \text{ ...... } (10)$$

particular eq^n of forced vibration.

Eq^n ~~of free~~ damped vi

W.K.T eq^n of free damped vibration is

$$y = ae^{-bt}\sin(\omega t-\alpha) \text{ ...... } (11)$$

The motion of forced vibration is combination of both particular eq^n of ~~for~~ external periodi force and eq^n of free damped vibration.

$$y = \text{eq}^n\ (10) + \text{eq}^n\ (11)$$

$$\boxed{y = \frac{F}{\sqrt{\mu^2p^2+(k-mp^2)^2}}\sin(pt-\alpha) + ae^{-bt}\sin(\omega t-\alpha)}$$